NONLINEAR CONICAL FLOW OF A GAS

(NELINEINYE KONICHESKIE TECHENIIA GAZA)

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In the paper by Giese and Cohn [1] the hodograph equations of irrotational steady conical flow of a gas are reduced to a canonical system of three equations for the velocity components u, v, w, the highest derivatives of which are in Laplacian form. The position of the ray, $\xi = x/z$, $\eta = y/z$, corresponding to the velocity u, v, w, is determined by derivatives of u, v, w, with respect to independent variables introduced by the authors. In using their equations to find approximate solutions, a number of difficulties appear. In the first place, it is not clear how the linearized theory will appear in their variables; secondly, ξ must be determined by derivatives of the approximate solution; thirdly, it is not clear whether every conical flow will be single-sheeted in the u, v plane. For these reasons, another method is proposed here. The quasi-linear differential equation of second order, $AF_{\xi\xi} + 2BF_{\xi\eta} + CF_{\eta\eta} = D$, where A, B, C, D are arbitrary functions depending on ξ, η, F, F_ξ, F_η, $AC - B^2 > 0$, is reduced to the canonical system of equations

$$\Delta\xi = \Phi_1, \quad \Delta\eta = \Phi_2, \quad \Delta u = \Phi_3, \quad \Delta v = \Phi_4, \quad \Delta w = \Phi_5$$
$$\Delta = \frac{\partial^2}{\partial\rho^2} + \frac{\partial^2}{\partial\sigma^2}, \quad u = F_\xi, \quad v = F_\eta, \quad w = F - \xi u - \eta v$$

Here the functions ϕ depend on ξ, ν, u, v, w and their first derivatives with respect to ρ, σ.

A method is given for obtaining the second approximation in the nonlinear theory of irrotational steady conical flow of a gas. A check on the accuracy of the method is given for the example of axisymmetric flow over a circular cone.

1. The differential equation for the irrotational nonlinear flow of a gas has the form

$$AF_{\xi\xi} + 2BF_{\xi\eta} + CF_{\eta\eta} = 0 \tag{1.1}$$

where

$$A = a^2(1+\xi^2) - (u - \xi w)^2$$
$$a^2 = a_0^2 - \tfrac{1}{2}(\varkappa - 1)(u^2 + v^2 + w^2 - W_0^2) \qquad (u = F_\xi)$$
$$B = (a^2 - w^2)\xi\eta + (u\eta + v\xi) w - uv \qquad (w = F - \xi u - \eta u)$$
$$C = a^2(1+\eta^2) - (v - \eta w)^2 \qquad \left(\xi = \frac{x}{z},\ \eta = \frac{y}{z}\right) \qquad (v = F_\eta)$$

a is the velocity of sound, κ is the ratio of specific heats, a_0, W_0 are respectively the velocity of sound and velocity at a certain point of the flow, u, v, w are the velocity components, the quantities ξ, η correspond to the coordinates x and y of points in the plane $z = 1$ of the xyz field.

If the projection of the velocity on the plane perpendicular to the radius-vector of the point in the xyz field is less than the velocity of sound, then equations (1.1) are of elliptic type. The coefficients A, B, C in (1.1) depend on the unknown function F, which makes it impossible to reduce equations (1.1) to a system of two first-order equations, and subsequently to canonical form according to Khristianovich's method [2]. In the following, a method is developed for reducing equations of type (1.1) to a canonical system. It appears that this had not been done for the general case (cf. for instance [3]). These results supplement Khristianovich's results.

Let us examine the equation

$$AF_{\xi\xi} + 2BF_{\xi\eta} + CF_{\eta\eta} = D \tag{1.2}$$

where A, B, C, D are arbitrary functions of ξ, η, F, F_ξ, F_η. Equation (1.2) is equivalent to the system

$$Au_\xi + B(u_\eta + v_\xi) + Cv_\eta = D \qquad dF = ud\xi + vd\eta \tag{1.3}$$

Here we make the substitution $w = F - \xi u - \eta v$ (all that follows below can also be carried through without this substitution, with no essential changes). We obtain

$$Au_\xi + B(u_\eta + v_\xi) + Cv_\eta = D, \qquad dw + \eta dv + \xi du = 0 \tag{1.4}$$

We introduce new independent variables ρ, σ defined by the relations

$$N = A\eta_\rho - B\xi_\rho + \sqrt{AC - B^2}\,\xi_\sigma = 0, \qquad R = A\eta_\sigma - B\xi_\sigma - \sqrt{AC - B^2}\,\xi_\rho = 0 \tag{1.5}$$

The quantities u, v, w, ξ, η are now considered to be functions of ρ, σ. The variables ρ, σ are the same as occur in Courant and Hilbert [3], but in the relations for ξ, η there given, the present author has succeeded in separating the factors N and R, which are linear in the derivatives. We note that ρ, σ have the meaning of "complex characteristics" [2]. That is, if we formally write the equation of the characteristics of (1.2) in the form

$$A\frac{d\eta}{d\tau}-(B^2+i\sqrt{AC-B^2})\frac{d\xi}{d\tau}=0$$

and go over from $\tau = \rho + i\sigma$, $\bar{\tau} = \rho - i\sigma$ to ρ, σ, then, after separating real and imaginary parts, we will obtain (1.5). It is easy to prove that equations (1.5) are invariant with respect to conformal transformation of the plane of the independent variables, $\rho + i\sigma = f(\rho^* + i\sigma^*)$, where f is an analytic function, ρ^* and σ^* are new variables. From (1.5) it follows that the Jacobian is

$$\xi_\rho\eta_\sigma-\xi_\sigma\eta_\rho=\frac{\sqrt{AC-B^2}}{A}(\xi_\rho^2+\xi_\sigma^2) \tag{1.6}$$

and can become zero only at isolated points. Going over to the variables ρ, σ in equation (1.4) and using (1.5) as well as the relations

$$X=-\frac{B}{A}N-\frac{\sqrt{AC-B^2}}{A}R=C\xi_\rho-B\eta_\rho-\sqrt{AC-B^2}\,\eta_\sigma=0$$
$$Y=\frac{\sqrt{AC-B^2}}{A}N-\frac{B}{A}R=C\xi_\sigma-B\eta_\sigma+\sqrt{AC-B^2}\,\eta_\rho=0 \tag{1.7}$$

we obtain a system equivalent to (1.2):

$$K=\xi_\rho u_\rho+\xi_\sigma u_\sigma+\eta_\rho v_\rho+\eta_\sigma v_\sigma-\frac{D}{\sqrt{AC-B^2}}(\xi_\rho\eta_\sigma-\xi_\sigma\eta_\rho)=0$$
$$L=w_\rho+\eta v_\rho+\xi u_\rho=0,\qquad N=A\eta_\rho-B\xi_\rho+\sqrt{AC-B^2}\,\xi_\sigma=0 \tag{1.8}$$
$$M=w_\sigma+\eta v_\sigma+\xi u_\sigma=0,\qquad R=A\eta_\sigma-B\xi_\sigma-\sqrt{AC-B^2}\,\xi_\rho=0$$

We form the relations

$$S=L_\rho+M_\sigma=\Delta w+\eta\Delta v+\xi\Delta u+\xi_\rho u_\rho+\xi_\sigma u_\sigma+\eta_\rho v_\rho+\eta_\sigma v_\sigma=0 \tag{1.9}$$
$$T=L_\sigma-M_\rho=\xi_\sigma u_\rho-\xi_\rho u_\sigma+\eta_\sigma v_\rho-\eta_\rho v_\sigma=0$$

The relations $S = T = 0$ will be equivalent to $L = M = 0$, if it is required that the condition $Ld\rho + Md\sigma = dw + \eta dv + \xi du = 0$ be fulfilled on the boundary of the region in which the solution is required.

In (1.8) we replace $L = M = 0$ by $S = T = 0$, and in $S = 0$ we replace the terms with first derivatives by the term which contains D in $K = 0$. Then we obtain the system

$$S=\Delta w+\eta\Delta v+\xi\Delta u+\frac{D}{\sqrt{AC-B^2}}(\xi_\rho\eta_\sigma-\xi_\sigma\eta_\rho)=0,\quad K=T=N=R=0 \tag{1.10}$$

Using (1.6) we form the combinations

$$K + iT = (\xi_\rho + i\xi_\sigma)(u_\rho - iu_\sigma) + (\eta_\rho + i\eta_\sigma)(v_\rho - iv_\sigma) - \frac{D}{A}(\xi_\rho + i\xi_\sigma)(\xi_\rho - i\xi_\sigma) = 0 \tag{1.11}$$

$$N + iR = A(\eta_\rho + i\eta_\sigma) - (B + i\sqrt{AC - B^2})(\xi_\rho + i\xi_\sigma) = 0 \tag{1.12}$$

If in relation (1.11) we replace the quantity $(\eta_\rho + i\eta_\sigma)/(\xi_\rho + i\xi_\sigma)$ by $(B + i\sqrt{AB - B^2})/A$ from (1.12) and separate real and imaginary parts in the resulting relation, we obtain two equations: $P = 0$, $Q = 0$, equivalent to $K = 0$, $T = 0$. With the remaining equations of (1.10) we have

$$\begin{gathered} P = Au_\rho + Bv_\rho + \sqrt{AC - B^2}v_\sigma - D\xi_\rho = 0 \\ Q = Au_\sigma + Bv_\sigma - \sqrt{AC - B^2}v_\rho - D\xi_\sigma = 0 \\ N = A\eta_\rho - B\xi_\rho + \sqrt{AC - B^2}\xi_\sigma = 0, \qquad R = A\eta_\sigma - B\xi_\sigma - \sqrt{AC - B^2}\xi_\rho = 0 \\ S = \Delta w + \eta\Delta v + \xi\Delta u + \frac{D}{\sqrt{AC - B^2}}(\xi_\rho\eta_\sigma - \xi_\sigma\eta_\rho) = 0 \end{gathered} \tag{1.13}$$

plus the boundary condition $dw + \eta dv + \xi du = 0$.

From the system (1.13) it is easy to go over to a system with second derivatives in Laplacian form. We form the relations

$$\begin{gathered} P_\rho + Q_\sigma = B\Delta v + A\Delta u - D\Delta\xi + \cdots = 0, \quad N_\rho + R_\sigma = A\Delta\eta - B\Delta\xi + \cdots = 0 \\ P_\sigma - Q_\rho = \sqrt{AC - B^2}\Delta v + \cdots = 0, \qquad N_\sigma - R_\rho = \sqrt{AC - B^2}\Delta\xi + \cdots = 0 \end{gathered} \tag{1.14}$$

The dots denote terms which contain no second derivatives. From (1.14) and $S = 0$, noting that $AC - B^2 \neq 0$, $A \neq 0$, we find

$$\Delta u = \Phi_1, \qquad \Delta v = \Phi_2, \qquad \Delta w = \Phi_3, \qquad \Delta\xi = \Phi_4, \qquad \Delta\eta = \Phi_5 \tag{1.15}$$

where the Φ depend on ξ, η, u, v, w and their first derivatives with respect to ρ, σ. The system (1.15) is equivalent to (1.13), if it is required that on the boundary the conditions

$$\begin{gathered} Pd\rho + Qd\sigma = Adu + Bdv + \sqrt{AC - B^2}(v_\sigma d\rho - v_\rho d\sigma) - Dd\xi = 0 \\ Nd\rho + Rd\sigma = Ad\eta - Bd\xi + \sqrt{AC - B^2}(\xi_\sigma d\rho - \xi_\rho d\sigma) = 0 \\ dw + \eta dv + \xi du = 0 \end{gathered} \tag{1.16}$$

be satisfied, since equations (1.14) for the functions $P + iQ$, $N + iR$ represent Cauchy-Riemann conditions.

For the case $D \equiv 0$, the system (1.14) may be obtained in a more symmetric form. We will show how this can be done for the case of ξ, η. Let N', R', X', Y' represent the quantities N, R, X, Y [cf. (1.5), (1.7)] divided by $\sqrt{AC - B^2}$. Form the relations

$$N_\sigma' - R_\rho' = 0, \qquad Y_\rho' - X_\sigma' = N_\rho' + R_\sigma' + EN' + FR' = 0 \qquad (1.17)$$

where E, F are functions which are bounded inside any closed sub-region of the region where the solution is required. If it is required for the condition $N'd\rho + R'd\sigma = 0$ to be satisfied on the boundary, then equations (1.17) are equivalent to $N' = R' = 0$. But $N_\sigma' - R_\rho' = \Delta\xi + \ldots = 0$, $Y_\rho' - X_\sigma' = \Delta\eta + \ldots = 0$.

The system (1.15) for $D \equiv 0$ has the form:

$$(1.18)$$

$$\Delta\xi = \left(\frac{B}{\sqrt{AC-B^2}}\right)_\sigma \xi_\rho - \left(\frac{B}{\sqrt{AC-B^2}}\right)_\rho \xi_\sigma + \left(\frac{A}{\sqrt{AC-B^2}}\right)_\rho \eta_\sigma - \left(\frac{A}{\sqrt{AC-B^2}}\right)_\sigma \eta_\rho$$

$$\Delta\eta = \left(\frac{B}{\sqrt{AC-B^2}}\right)_\rho \eta_\sigma - \left(\frac{B}{\sqrt{AC-B^2}}\right)_\sigma \eta_\rho + \left(\frac{C}{\sqrt{AC-B^2}}\right)_\sigma \xi_\rho - \left(\frac{C}{\sqrt{AC-B^2}}\right)_\rho \xi_\sigma$$

$$\Delta u = \left(\frac{B}{\sqrt{AC-B^2}}\right)_\sigma u_\rho - \left(\frac{B}{\sqrt{AC-B^2}}\right)_\rho u_\sigma + \left(\frac{C}{\sqrt{AC-B^2}}\right)_\sigma v_\rho - \left(\frac{C}{\sqrt{AC-B^2}}\right)_\rho v_\sigma$$

$$\Delta v = \left(\frac{B}{\sqrt{AC-B^2}}\right)_\rho v_\sigma - \left(\frac{B}{\sqrt{AC-B^2}}\right)_\sigma v_\rho + \left(\frac{A}{\sqrt{AC-B^2}}\right)_\rho u_\sigma - \left(\frac{A}{\sqrt{AC-B^2}}\right)_\sigma u_\rho$$

$$\Delta w = -\eta\Delta v - \xi\Delta u$$

plus the boundary condition (1.16). This system is invariant with respect to conformal transformation of the plane of the independent variables.

We will note, for later reference, that the system (1.8) is equivalent to the system

$$S = 0, \qquad L = M = N = R = 0 \qquad (1.19)$$

where $S = 0$ is taken in its form in system (1.10).

2. We shall apply the method of successive approximations to the equation of conical flow of a gas (1.1), transformed to the system (1.8). In the right-hand sides of (1.18), and in the coefficients of the derivatives in the boundary conditions (1.16), we put the values corresponding to the first approximation, for which we use the ordinary linearized theory of flow around conical bodies. This gives a system of Poisson equations for the second approximation, etc.

Let us find the solution of (1.18), or the equivalent system (1.13), which corresponds to a uniform flow $u = v = 0$, $w = w_0$ (axis Oz is in the flow direction). Putting $u = v = 0$, $w = w_0$ in (1.13) we find that $P = Q = S = 0$ are satisfied, while the equations $N = R = 0$ have the solutions which are obtained by first transforming to polar coordinates in the planes ξ, η, and ρ, σ, according to the equations

$$\sqrt{M_0^2 - 1}\,\xi = r\cos\theta, \quad \sqrt{M_0^2 - 1}\,\eta = r\sin\theta, \quad \rho = \alpha\cos\beta, \quad \sigma = \alpha\sin\beta \quad (2.1)$$

where M_0 is the free stream Mach number

$$r = \frac{2\alpha}{1+\alpha^2}, \qquad \theta = \beta$$

The linearized theory is obtained if, in the right-hand sides of the system (1.18), in A, B, C, ξ, η and their derivatives, we put the values

$$r_{(0)} = \frac{2\alpha}{1+\alpha^2}, \qquad \theta_{(0)} = \beta, \qquad u = v = 0, \qquad w = w_0 \qquad (2.2)$$

The first derivatives of u, v are left in the exact form, since they belong to the functions which are to be found; that is, the equations for u, v, corresponding to the linearized theory, contain not only Laplacians but also first derivatives of the unknown functions.

Actually, we shall put the values (2.2) only in A, B, C, in (1.18) as well as (1.16). Then, proceeding in a direction opposite to that which led to (1.18), that is going from (1.18) to (1.13) ... (1.8), and, finally, to (1.19), we can establish that $u_{(1)}$, $v_{(1)}$, $w_{(1)}$, $\xi_{(1)}$, $\eta_{(1)}$ satisfy (1.19), when A, B, C correspond to (2.2).

Since in the linearized theory the outer boundary of the conical flow is the Mach cone, we can again take the solutions of the equations $N = 0$, $R = 0$ to be $r_{(1)} = 2\alpha/(1+\alpha^2)$, $\theta_{(1)} = \beta$, which satisfy those equations.

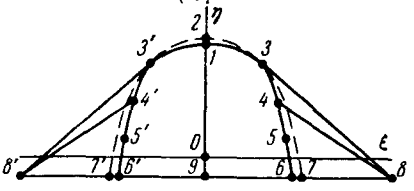

Fig. 1.

In (1.19) we are left with the equations

$$S = \Delta w_{(1)} + \eta_{(1)}\Delta v_{(1)} + \xi_{(1)}\Delta u_{(1)} = 0$$

$$L = w_{(1)\rho} + \eta_{(1)}v_{(1)\rho} + \xi_{(1)}u_{(1)\rho} = 0$$

$$M = w_{(1)\sigma} + \eta_{(1)}v_{(1)\sigma} + \xi_{(1)}u_{(1)\sigma} = 0$$

It is easy to prove that the solution of this system is given by three harmonic functions, which are related by the equation

$$d\,[u_{(1)} + iv_{(1)}] = -\tfrac{1}{2}\Big[\tau d f_{(1)} + \tfrac{1}{\tau}\,\overline{d f_{(1)}}\Big] \tag{2.3}$$

where $\tau = a\,e^{i\beta}$, $\bar{\tau} = a\,e^{-i\beta}$, $f_{(1)} = m\,[w_{(1)} + is]$ is an analytic function of τ, $m = \sqrt{M_0^2 - 1}$. Indeed, $S = 0$ is satisfied since $\Delta u_{(1)} = \Delta v_{(1)} = \Delta w_{(1)} = 0$, and $d w_{(1)} + \eta_{(1)} d v_{(1)} + \xi_{(1)} d u_{(1)} = 0$, which is equivalent to $L = M = 0$, appears as the real part of equation (2.3), multiplied by $e^{-i\beta}$. Since $r_{(1)} = r_{(0)}$, $Q_{(1)} = Q_{(0)}$, then if we replace values with index (1) by values with index (0) wherever possible in equations (1.16) and (1.18), we find that the linearized theory appears as the solution of (1.18) with the specified right-hand side, and the τ-plane is the plane introduced by Busemann [4].

The invariance of (1.18) with respect to conformal transformation (of the plane of the independent variables) makes it possible to look for solutions in any convenient region of the $\rho\,\sigma$-plane. We shall examine separately the cases in which the region occupied by the conical flow in the ξ, η plane is singly connected and doubly connected. For the case of the singly connected region we shall take the example of flow over the upper portion of a plane delta wing with supersonic edges, at angle of attack δ [5].

The envelope of the Mach cones with vertices along the leading edge is defined by the planes *8-3*, *8'-3'*, and the arc of the Mach cone *3-1-3'* (Fig. 1). After the straight lines *3-8*, *3'-8'*, come Prandtl-Meyer fans, followed by regions of uniform flow. The region of conical flow is bounded by a weak, unbroken shock wave, *7-3-2-3'-7'*, and the plane of the wing, *8-7-7'-8'*. This shock lies near the Mach cone *3-1-3'*, the arc of the curvilinear characteristics *3-4* and *3'-4'* of the Prandtl-Meyer flow, the straight characteristics *4-5*, *4'-5'*, and the arcs of the Mach cones *5-6* and *5'-6'* of the expanded flow. The author assumes that $AC - B^2 > 0$ in this region (in those cases for which the angle at the vertex of the wing is substantially different from π). In the linearized formulation, the region of conical flow for $\xi > 0$ has the form shown in Fig. 2.

For the region of the variables $\rho\,\sigma$ in the exact solution we take the plane which is obtained by mapping the ξ, η plane of the linearized problem on to the ρ, σ plane according to the equations $r = 2a/(1 + a^2)$, $\theta = \beta$. The second approximation is also sought in that plane. (For the actual determination of the second approximation, this region may be mapped on to a circle or on to some other region for which Green's function is known). In our example, the region in the ρ, σ plane is a quadrant of the unit circle.

For finding the second approximation in the case of a doubly connected region (a body entirely enclosed by the shock wave) the region of the linearized problem (Fig. 3) in the $\xi\eta$ plane is also transformed into the $\rho\sigma$ plane according to the equation $r = 2a/(1 + a^2)$, $\theta = \beta$, and this region is transformed into a ring, whose internal radius we will fix. Correspondingly the linearized solution is transformed to new variables. We will seek the second approximation in a ring whose inner radius coincides with that of the first approximation, while the outer radius, which is not at first known, will be found as part of the solution of the second approximation. The difference in these radii for a circular cone is $O(\epsilon)$, where ϵ is the half-angle of the cone, and is illustrated in Fig. 4, where R_0 is the fixed radius, R_1 is the radius in the first approximation, and R_2 is the radius in the second approximation.

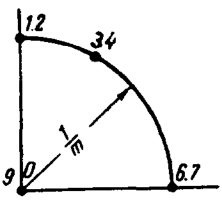

Fig. 2.

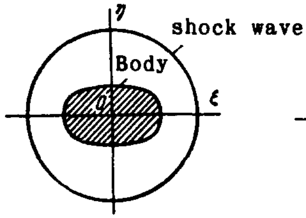

Fig. 3.

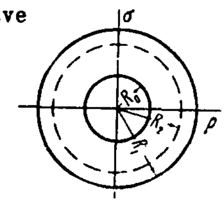

Fig. 4.

The boundary conditions at the shock wave, which express the continuity of the velocity component tangent to the shock surface, and the condition for the normal component for an oblique shock, have the form

$$w - w_0 + \eta v + \xi u = 0$$

$$\left(\frac{dr}{d\theta}\right)^2\left[1 + \frac{w_0 - v}{a_0} M_0 \frac{\varkappa + 1}{2}\right] = r^2\left[r^2 - 1 - \left(1 + \frac{r^2}{m^2}\right)\frac{w_0 - w}{a_0} M_0 \frac{\varkappa + 1}{2}\right] \quad (2.4)$$

where $(0, 0, w)$ and (u, v, w) are the velocity components before and after the shock, ξ, η are coordinates of the shock surface, $r = m\sqrt{\xi^2 + \eta^2}$, $\tan\theta = \eta/\xi$, κ is the adiabatic exponent, $m = \sqrt{M_0^2 - 1}$, M_0 and a_0 are the Mach number and velocity of sound ahead of the shock.

For bodies which are entirely enclosed by the shock wave, neglecting $(dr/dt)^2$ and putting $r = 1$ (Mach cone), for the second approximation we obtain

$$w_{(2)} - w_0 + \eta_{(1)} v_{(2)} + \xi_{(1)} u_{(2)} = 0, \qquad r_{(2)} - 1 + \frac{M_0^4}{m^2}\frac{\varkappa + 1}{4}\frac{w_{(2)} - w_0}{w_0} = 0 \quad (2.5)$$

For bodies which extend through the Mach cone it is also possible to make a linearization (2.4) appropriate to the specific problem. The condition which states that the velocity component normal to the body is zero has the form

$$Z_\xi (u - \xi w) + Z_\eta (v - \eta w) = 0 \tag{2.6}$$

where $Z(\xi, \eta) = 0$ is the equation of the body contour. This may be linearized if all the terms are taken to be a first approximation plus a correction, and terms containing squares or higher powers of the corrections are neglected.

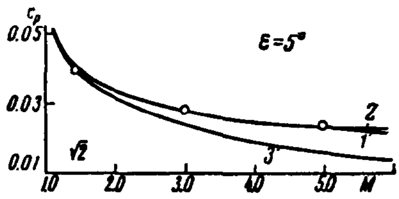

Fig. 5.

On the Mach cone, $(AC-B^2)_{(1)}$ becomes zero and its surface becomes singular in the determination of the second approximation, though this circumstance does not require the introduction of special methods, since the author finds that already $(AC - B^2)_{(2)} \neq 0$. The author will not dwell on the series of technical difficulties that may appear in the working out of the second approximation. For example, $(AC - B^2)_{(1)} < 0$ on *3-6* (Fig. 2); this results from the circumstance that the boundary conditions are taken on the Mach cone of the undisturbed flow. This difficulty can be avoided if the boundary conditions are taken on the Mach cone of the expanded flow, which may be done without going beyond the limits of accuracy of the linearized theory.

3. To evaluate the accuracy of the second approximation, it was worked out for axisymmetric flow over a circular cone. It may be shown that the solution for a cone depends only on a [cf. (2.1)].

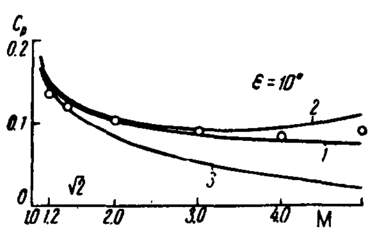

Fig. 6.

If the radial velocity component is denoted by λ, the system (1.19) becomes

$$w'' + \frac{1}{\alpha} w' + \frac{r}{m}\left(\lambda'' + \frac{1}{\alpha}\lambda' - \frac{1}{\alpha^2}\lambda\right) = 0$$

$$w' + \frac{r}{m}\lambda' = 0 \tag{3.1}$$

$$\alpha r' - r\left[1 + \frac{r^2}{m^2} - \frac{(\lambda - wr/m)^2}{a^2}\right]^{1/2} = 0$$

Putting w' from the second equation of (3.1) into the first, and using the third equation, we find

$$\alpha w' = \frac{r}{m}\left[1 + \frac{r^2}{m^2} - \frac{(\lambda - wr/m)^2}{a^2}\right]^{-1/2}\lambda$$

$$\alpha\lambda' = -\left[1 + \frac{r^2}{m^2} - \frac{(\lambda - wr/m)^2}{a^2}\right]^{-1/2}\lambda \tag{3.2}$$

$$\alpha r' = r\left[1 + \frac{r^2}{m^2} - \frac{(\lambda - wr/m)^2}{a^2}\right]^{1/2}$$

The system (3.2) is essentially equivalent to (1.18), since the linearized solution is obtained if the squared bracket and r in the right-hand sides of (3.2) are replaced by their values corresponding to the uniform flow, $u = v = 0$, $w = w_0$, $r = 2a/(1 + a^2)$.

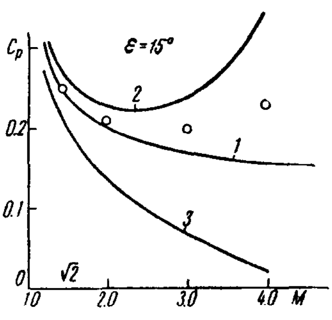

Fig. 7.

If in the right-hand side of (3.2) we put values corresponding to slender body theory rather than the linearized theory, which we do to simplify the calculations,

$$r_{(1)} = \frac{2\alpha}{1 + \alpha^2}, \qquad \lambda_{(1)} = w_0 \frac{m\varepsilon^2}{2}\left(\frac{1}{\alpha} - \alpha\right), \qquad w_{(1)} = w_0(\varepsilon^2 \ln \alpha + 1) \tag{3.3}$$

then we obtain the system of the second approximation, which can be solved by integration. Satisfying the flow condition on the body for $a = 1/2m\epsilon$, $\lambda_{(2)} = w_{(2)} \tan \epsilon$, the condition (2.5) at the shock gives for a_1, determining for the outer radius of the ring, the value

$$\alpha_1 = 1 - k\varepsilon - l\varepsilon^2 \ln \frac{m\varepsilon}{2} - \cdots \tag{3.4}$$

where

$$k = 2M_0\left[\frac{\pi}{2} - 1 - \frac{4}{m^2}\left(1 + \frac{\varkappa - 1}{2}M_0^2\right)\int_0^1 \frac{\alpha^2 \ln \alpha d\alpha}{(1+\alpha^2)(1-\alpha^2)}\right]^{1/2} \qquad (3.5)$$

$$l = -\frac{\varkappa+1}{4}\frac{M_0^4}{m^3}\frac{1}{k}$$

The term containing l takes into account the shock position. On the shock, $r_{(2)} = 1 + O(\epsilon^3 \ln 1/2m\epsilon)$, which is not correct; but the influence of the shock on the velocity components at the body within the present theory is $O(\epsilon^6 \ln\epsilon)$, and it is not necessary to take it into account in the solution of other problems.

The pressure coefficient on the cone surface is given by the expression

$$\begin{aligned} C_p = &-\varepsilon^2 - 2\varepsilon^2 \ln\frac{m\varepsilon}{2} + \frac{2}{m}\varepsilon^3 \ln\frac{m\varepsilon}{2} + 2\left(\frac{1}{3m} - \frac{1}{3} + \frac{m}{4} + \frac{5}{6}\frac{M_0^2}{m}\right)\varepsilon^3 + \\ &+ m^2\varepsilon^4 \ln^2\frac{m\varepsilon}{2} + 2\left(M_0^2 - \frac{1}{m^2} - 1\right)\varepsilon^4 \ln\frac{m\varepsilon}{2} + \\ &+ \left\{\frac{M_0^2}{4} - \frac{2}{3m^2} - \frac{1}{2} - \frac{5}{3}\frac{M_0^2}{m^2} + M_0^2\left[\frac{2\ln 2}{m^2}(2 + (\varkappa-1)M_0^2) + \right.\right. \\ &\left.\left. + \frac{\pi - 1 - \ln 2}{2}\right]\right\}\varepsilon^4 + O\left(\varepsilon^5 \ln^2\frac{m\varepsilon}{2}\right) \end{aligned} \qquad (3.6)$$

This expression does not agree with that given by Broderick, [7], which is easily explained, since the two solutions have been obtained along different lines. Calculations using equation (3.6) show that it gives somewhat more accurate results than Broderick's formula for the second approximation.

Figures 5, 6, 7 for cone angles ϵ = 5, 10, 15° are taken from Broderick [7]; the curves marked *1* correspond to the numerical solution, curves 2 are Broderick's second approximation, curves *3* are the slender body theory; the points are the values of C_p computed from equation (3.6).

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